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ABSTRACT

Horizontal density gradients are often present in estuaries. These gradients exert a torque which can induce a flow that may aid in reducing the residence time. Using simple models, the influence of a horizontal density gradient on the residence time of a shallow, well-stirred lagoon was investigated. Under steady-state conditions and with scaling taken from the southeast Florida lagoons (Biscayne Bay and Card Sound), it was found that density-induced motions do not contribute substantially to flushing the lagoon waters. However, as an estuary's depth increases or, if the horizontal density gradient is large, density-induced flow can be significant. A graphic approach is presented which can be used to determine density-induced residence time from knowledge of the horizontal density gradient and the water depth.

KEYWORDS

Estuaries; Density; horizontal density gradients; density flows; Tides; Water Flow; currents; Water Pollution; residence time; Water Quality

DENSITY-INDUCED MOTIONS IN SHALLOW LAGOONS

by

DONALD R. JOHNSON AND THOMAS N. LEE

INTRODUCTION

Neither the residence times nor the dominant flushing mechanisms are established with certainty for the shallow intracoastal lagoons which occur along southeast Florida. To guide observational programs, however, it is important to develop a feeling for the kinds of mechanisms which may be important and to eliminate the others from principal consideration. In this study, two cases are investigated in which the horizontal density gradient plays the major role in inducing water motions.

In the shallow intracoastal waters of southeast Florida, tide and wind mixing are of sufficient strength to maintain vertical near-homogeneity in the water column. However, because of river runoff and because of seasonal insolation, together with evaporation and precipitation in the shallow water, a horizontal density gradient is normally present between the interior portions of these lagoons and the exterior portions which are partially open to the Florida coastal waters. The objective of this study is to determine if these horizontal density gradients induce water motions of sufficient intensity and direction to influence the residence time within the lagoon.

The authors have chosen to investigate, first, the steady condition in which both wind and tidal mixing are assumed to be constant over a complete tidal cycle, and second, the time-dependent influence due to the reduction of mixing during slack tide. The mathematical model will be developed, scaled, and analyzed for a simple steady state case, which neglects the input of mean wind momentum, and then applied to a more complex case which includes the effect of density-induced accelerations during slack tide.

DEVELOPMENT OF THE SIMPLE CASE

The intracoastal lagoons of southeast Florida are aligned with their major axis running in a nearly north-south direction. The coordinate system for this model (Fig. 1) will have the positive y-axis pointing north along the major axis, the positive x-axis pointing east toward the exterior of the lagoon, and the positive z-axis pointing up. The origin of this coordinate system will lie at the surface and at the inside of the lagoon.

Since the isopycnals are, to first order, aligned along the major axis (Lee and Rooth, 3, 4), the simple steady state case will neglect variations in the y-direction. An implication which runs throughout this work, and which helps to simplify a complex situation, is that only interior portions of the lagoons are delt with. By assuming that the relatively thin side wall boundary layer acts only to maintain the boundary characteristics of a more slowly varying interior, the larger and more complicated gradients within the side wall boundary layer can be neglected. The momentum equations, continuity equation, and equation for the conservation of density are written as:

$$-fv - Au_{zz} - A_h u_{xx} = -\alpha P_x \qquad (1)$$

$$fu - Av_{zz} - A_h v_{xx} = 0 \qquad (2)$$

$$g = -\alpha P_z \qquad (3)$$

$$u_x + w_z = 0 \qquad (4)$$

$$u\rho_x + w\rho_z - K\rho_{zz} - K_h\rho_{xx} = 0 \qquad (5)$$

where (u,v,w,ρ,P,g) correspond to commonly used definitions for velocity components, density, pressure, and the acceleration of gravity, respectively; f is the Coriolis parameter, and α is the specific volume; A and K are vertical

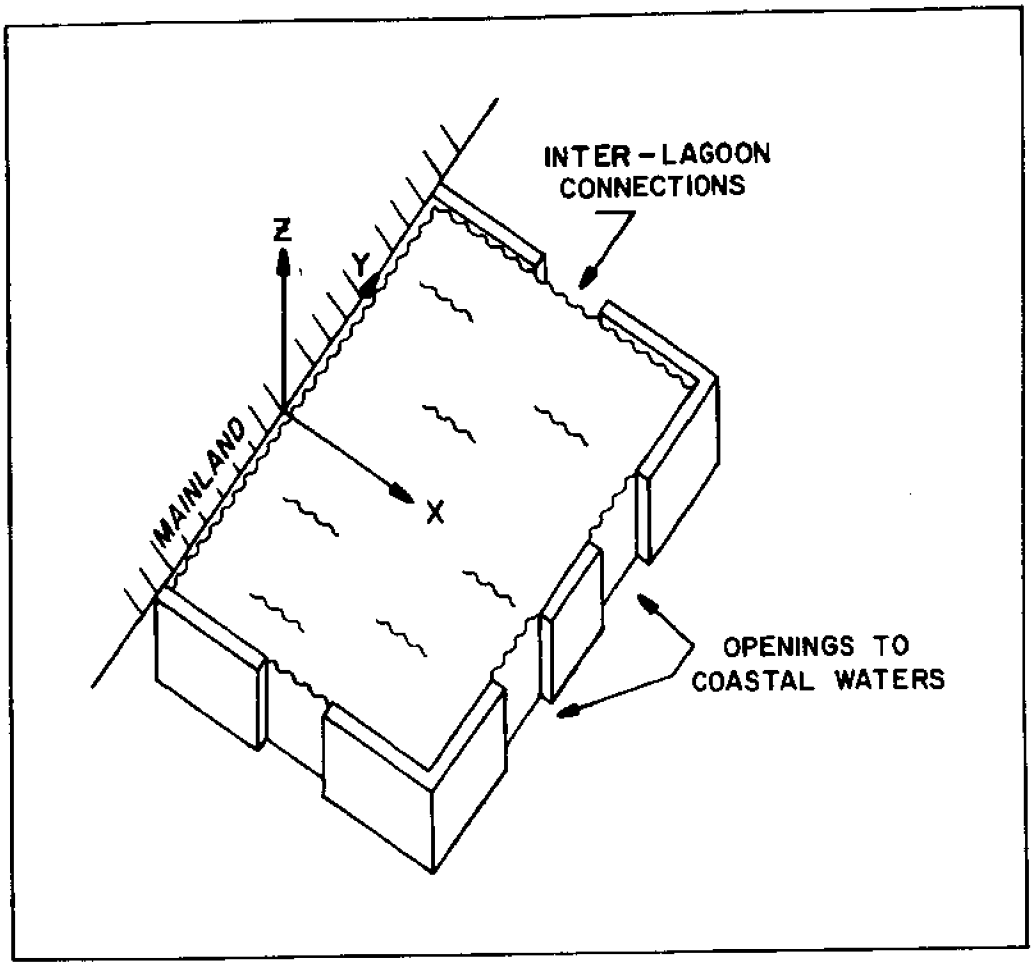

Figure 1. Schematic of intralagoon coordinate system

Austauch and diffusion coefficients; and A_h and K_h are x-component horizontal Austauch and diffusion coefficients, respectively.

Assume that the variables can be broken into a vertically averaged part plus a vertically dependent part:

$$(u,v,w,\rho) = \overbrace{(\bar{u},\bar{v},\bar{w},\bar{\rho})}^{\text{function of x only}} + \overbrace{(u',v',w',\rho')}^{\text{function of x and z}}. \tag{6}$$

This will be useful since the vertical variation of ρ is small and, therefore, can be treated as an anomaly. Three simplifying assumptions act to restrict the vertically averaged current within the lagoon: (1) steady state, (2) no sources or sinks of current at the x-boundaries, and (3) no y-gradients. Then,

$$\int_{-H}^{0} u dz = \int_{-H}^{0} w \, dz = 0.$$

Hence, $\bar{u} = \bar{w} = 0$, and

$$u = u'(x,z)$$

$$v = \bar{v}(x) + v'(x,z)$$

$$w = w'(x,z)$$

$$\rho = \bar{\rho}(x) + \rho'(x,z)$$

but $\bar{\rho} \sim O(1)$ and $\rho' \sim O(10^{-3})$ or less, so that $\bar{\rho} \gg \rho'$.

Expand Eqs. 1, 2, 4, and 5 in terms of Eq. 6 and take the vertical derivative of Eqs. 1 and 2:

$$-fv'_z - Au'_{zzz} - A_h u'_{zxx} = -\alpha P_{xz} - \alpha_z P_x \simeq \alpha g \bar{\rho}_x \tag{7}$$

$$fu'_z - Av'_{zzz} - A_h v'_{zxx} = 0 \tag{8}$$

$$u'_x + w'_z = 0 \tag{9}$$

$$u'\bar{\rho}_x + w'\rho'_z - K\rho'_{zz} - K_h \bar{\rho}_{xx} = 0 \tag{10}$$

where it has been assumed that $\bar{\rho} >> \rho'$, $\bar{\rho}_x >> \rho'_x$, and $\bar{\rho}_{xx} >> \rho'_{xx}$ and that the buoyancy effect of the pressure term, $\alpha_z P_x$, is negligible (since we are assuming near-homogeneity in the vertical, there will be little influence of buoyancy on vertical motions). It will be useful, at this time, to define stream functions for the current field. From Eq. 9:

$$\left.\begin{aligned} \psi'_z &\equiv -u' \\ \psi'_x &\equiv w' \end{aligned}\right\}. \tag{11}$$

SCALING

It is expected that the heavier water from one side of the lagoon will flow along the bottom toward the region of lighter water and that the lighter water will return along the surface; hence, the vertical profiles of current and density will be nearly modal in form with the function $\cos \frac{\pi(z+m)}{H}$. Then, vertical derivatives can be scaled as $\frac{\partial}{\partial z} \sim \frac{\pi}{H}\frac{\partial}{\partial z^*}$ where the asterisk signifies a scaled variable. Horizontal derivatives in momentum will be scaled as $\frac{\partial}{\partial x} \sim \frac{\pi}{L}\frac{\partial}{\partial x^*}$ since the width of the basin essentially represents a half wave length for cross basin momentum. Horizontal derivatives in ρ will be avoided by carrying along the value of ρ_x. Horizontal derivatives in ρ_x, however, will be scaled as $\frac{1}{L}$. The variables are scaled as follows:

$$\psi'_z = U\psi^*_z$$

$$v' = Vv^*$$

$$w' = Ww^*$$

$$\rho'_z = \rho_z \; \rho^*_z$$

$$\bar{\rho}'_x = \rho_x \; \bar{\rho}^*_x$$

$$\bar{\rho}'_{xx} = (\rho_x/L) \; \bar{\rho}^*_{xx}$$

Applying these scales and using the stream function definitions, Eq. 11, then Eqs. 7, 8, and 10 are rewritten as:

$$-\left(\frac{\pi f V}{H}\right) v^*_z + \left(\frac{\pi^3 A U}{H^3}\right) \psi^*_{zzzz} + \left(\frac{\pi^3 A_h U}{HL^2}\right) \psi^*_{zzxx} = (\alpha g \rho_x) \bar{\rho}^*_x \tag{12}$$

$$-\left(\frac{\pi f U}{H}\right) \psi^*_{zz} - \left(\frac{\pi^3 A V}{H^3}\right) v^*_{zzz} - \left(\frac{\pi^3 A_h V}{HL^2}\right) v^*_{xxz} = 0 \tag{13}$$

$$-(U\rho_x)\; \psi^*_z \; \bar{\rho}^*_x + (W\rho_z)\; \psi^*_x \rho^*_z - \left(\frac{\pi K \rho_z}{H}\right) \rho^*_{zz} = \left(\frac{K_h \rho_x}{L}\right) \bar{\rho}^*_{xx}. \tag{14}$$

To avoid cluttering the equations, the asterisks over the differentiating variables have been dropped.

Dividing Eqs. 12 and 13 by their vertical friction term scales and dividing Eq. 14 by the vertical diffusion term scale yields:

$$-\left(\frac{H^2 f}{\pi^2 A}\right)\left(\frac{V}{U}\right) v^*_z + \psi^*_{zzzz} + \left(\frac{A_h H^2}{AL^2}\right) \psi^*_{zzxx} = \left(\frac{\alpha g H^3 \rho_x}{\pi^3 A U}\right) \bar{\rho}^*_x \tag{15}$$

$$-\left(\frac{H^2 f}{\pi^2 A}\right)\left(\frac{U}{V}\right) \psi^*_{zz} - v^*_{zzz} - \left(\frac{A_h H^2}{AL^2}\right) v^*_{zxx} = 0 \tag{16}$$

$$-\left(\frac{HU}{\pi K}\right)\left(\frac{\rho_x}{\rho_z}\right) \psi^*_z \bar{\rho}^*_x + \left(\frac{HW}{\pi K}\right) \psi^*_x \rho^*_z - \rho^*_{zz} = -\left(\frac{K_h H}{K \pi L}\right)\left(\frac{\rho_x}{\rho_z}\right) \bar{\rho}^*_{xx} \tag{17}$$

An estimation of the vertical Austauch coefficient can be made if it is assumed that the entire water column lies within the friction layer and that the mixing length increases linearly from the bottom. Then, an average value for A is given as:

$$A \sim \frac{ku_*}{H}\int_{-H}^{0} zdz = \frac{ku_*H}{2} ,$$

where k is the Karman coefficient ~ 0.4 and u_* is the characteristic friction speed estimated from the total stress on the surface and on the bottom, given by:

$$u_* \sim \left(\frac{\tau}{\rho}\right)^{1/2}.$$

It is assumed that, due to strong mixing, the Prandtl number $P_r = \frac{A}{K} \sim 0(1)$. Elder (3) has determined the effective horizontal dispersion coefficients in the turbulent shear flow (logarithmic) in a wide channel where the main flow is aligned along the channel axis. This compares well with the situation in the south Florida lagoons where the tidal flow is mainly in the longitudinal direction. Elder found that the horizontal dispersion coefficient in the longitudinal direction (y) was given by the relationship, 5.9 u_*H; in the cross channel direction (x), by the relationship, 0.2 u_*H. These relationships were also used in a numerical model of Biscayne Bay, Florida, by the Department of Coastal and Oceanographic Engineering at the University of Florida (Dean, 2). Assuming that the horizontal Prandtl number is also of 0(1), then $K_h = A_h \sim 0.2\ u_*H$. Therefore, $A_h \sim 0(A)$ and $K_h \sim 0(K)$.

The following scales are selected as characteristic of the intracoastal lagoons of southeast Florida, more specifically, Card Sound, Florida:

$$H = 3 \times 10^2 \text{ cm}$$

$$L = 5 \times 10^5 \text{ cm}$$

$$\rho_x = 5 \times 10^{-9} \text{ gms cm}^{-4}$$

$$u_* = 0.5 \text{ cm sec}^{-1}$$

$$f = 5 \times 10^{-5} \text{ sec}^{-1}$$

$$A = .2u_*H = 30 \text{ cm}^2 \text{ sec}^{-1}$$

$$A_h \sim 0(A)$$

$$K_h \sim 0(K)$$

From **Eq. 16, the ratio of friction terms provides:**

$$\frac{A_h}{A} \frac{H^2}{L^2} \sim 0(10^{-7}).$$

Then the significant balance in **Eq. 16 is:**

$$\frac{U}{V} \sim \frac{\pi^2 A}{H^2 f} \equiv E_k$$

where E_k is the Ekman number. Its value is $E_k \sim 0(70)$. This is a very high value implying that the Coriolis term is negligible in comparison to frictional stirring and that V is negligible compared to U. With this scaling, the ratio of the Coriolis to friction terms in **Eq. 15 is of the order** $\frac{1}{E_k^2} \sim 0(10^{-4})$. Therefore, the significant balance must be between the friction term and the pressur term. This balance provides the relationship:

$$U \sim \frac{\alpha g H^3 \rho_x}{\pi^3 A} \sim 0(10^{-1}) \text{ cm sec}^{-1}.$$

With the help of the continuity relationship to find $W \sim \frac{H}{L}U$ and with the relationship above given for U, the equation for density, **Eq. 17, can now be** investigated:

$$-(Ra)\left(\frac{\rho_x}{\rho_z}\right) \Psi_z^* \bar{\rho}_x^* + (Ra)\left(\frac{H}{L}\right) \Psi_x^* \rho_z^* - \rho_{zz}^* = -\left(\frac{K_h}{K}\right)\left(\frac{H}{\pi L}\right)\left(\frac{\rho_x}{\rho_z}\right) \bar{\rho}_{xx}^*$$

where Ra, the Rayleigh number, is defined as:

$$Ra \equiv \frac{\alpha g H^4 \rho_x}{\pi^4 A K} \sim 0(.5).$$

Since $\frac{H}{L} \sim 0(10^{-3})$, the vertical advection term is small compared to the vertical diffusion term and can be neglected. Comparing the horizontal advection term to the horizontal difussion term gives a ratio of the two terms as $\left(\frac{\pi L}{H}\right)\left(\frac{K}{K_h}\right)Ra$. In the interior of the lagoon, where $K_h \sim 0(K)$, this ratio is large. Then, in this region, horizontal diffusion is neglected in comparison to horizontal advection. Conversely, in the side wall boundary layers, the horizontal diffusion term is expected to be large. However, according to the initial assumptions, this layer is thin and negligible in formulating the interior dynamics. The remaining balance between horizontal advection and vertical diffusion provides:

$$\frac{\rho_z}{\rho_x} \sim 0(Ra)$$

which gives a value for ρ_z of the order $0(10^{-9})$ gms cm^{-4}.

With the values now available, an estimate can be made of the rate of change of density across the basin due to the flux of mass from the high density side to the low density side. The total flux through a unit area normal to the direction of flow is determined by the average over the depth:

$$F' = \frac{1}{H}\int_{-H}^{0} u'\rho' dz.$$

In scaling notation, an estimate of the vertically average flux is:

$$F = U\rho_z \frac{H}{\pi} \simeq 0(10^{-8}) \text{ gms } cm^{-2} sec^{-1}.$$

The rate of change of density is governed by the convergence of this flux:

$$F_x \approx \frac{\pi F}{L} \approx 0(6 \times 10^{-14}) \text{ gms cm}^{-3} \text{ sec}^{-1}.$$

Finally, the time needed to eliminate the horizontal density gradient (assumed constant across the lagoon) is defined as the residence time scale Tr:

$$\text{Residence time scale} \equiv Tr = L\rho_x/F_x \approx 10^3 \text{ years.} \tag{18}$$

It is evident that this process, for the scales represented above, is insufficient as a mechanism for reducing the residence time. Since Eq. 18 is a complicated function of the basic input parameters, particularly of u_*, H, and ρ_x, there exists a range of values for which this mechanism is important. However, before investigating this range, the equations will be solved and the scales checked for validity.

SOLUTION OF THE SIMPLE CASE

Using the results of scaling to eliminate unnecessary terms, the equations to be solved are:

$$\Psi^*_{zzzz} = \bar{\rho}^*_x \tag{19}$$

$$\rho^*_{zz} = -\Psi^*_z\bar{\rho}^*_x \tag{20}$$

with the following boundary conditions for Eq. 19:

(I)	$\Psi^*(0) = 0$	at $z = 0$
(II)	$\Psi^*(-1) = 0$	at $z = -1$
(III)	$\Psi^*_z(-1) = 0$	at $z = -1$
(IV)	$\Psi^*_{zz}(0) = 0$	at $z = 0$.

Boundary conditions (I) and (II) result from the condition [illegible] in the χ-direction. Boundary condition (III) stipulates "no-slip" on the bottom and (IV) assumes no wind stress at the sea surface.

The two boundary conditions for the density equation, Eq. 20, are:

$$\text{(V)} \qquad \rho^*_z(-1) = 0$$

$$\text{(VI)} \qquad \int_{-1}^{0} \rho^* dz = 0$$

where (V) stipulates that there is no mass flux across the sea bottom and (VI) comes from the basic definition of ρ'. Because of the symmetry involved, it also turns out that $\rho^*_z(0)=0$.

The solutions for Eqs. 19 and 20 are found by simple integration of these equations and the application of boundary conditions (I) through (VI). The solutions are:

$$\left.\begin{aligned} \psi^* &= \frac{\bar{\rho}^*_x}{8}\left(\frac{z^4}{3} + \frac{z^3}{2} - \frac{z}{6}\right) \\ \text{or} \qquad & \\ u^* &= -\frac{\bar{\rho}^*_x}{8}\left(\frac{4}{3} z^3 + \frac{3}{2} z^2 - \frac{1}{6}\right) \end{aligned}\right\} \qquad (21)$$

$$\rho^* = -\frac{(\bar{\rho}^*_x)^2}{8}\left(\frac{z^5}{15} + \frac{z^4}{8} - \frac{z^2}{12} + \frac{1}{72}\right) \qquad (22)$$

with conversion scales of:

$$u' = \left(\frac{\alpha g H^3 \rho_x}{A}\right) u^* \text{ and } \rho' = \left(\frac{\alpha g H^5 \rho_x^2}{AK}\right)\rho^*.$$

The factors of π have been removed since they specify the shape of the profile. This shape is now determined by the boundary conditions. Since $\bar{\rho}_x$ has been assumed a constant throughout the interior of the lagoon, w' is zero; and, hence, $\psi^*_x = 0$.

A plot of normalized u* and ρ^* versus normalized depth $(\frac{z}{H})$ is given in

Fig. 2, where u* and ρ* have been normalized by their respective maximum values, u*(0) and ρ*(0), and z has veen normalized by H. To convert back to u' and ρ', multiply the graph values of u*(z)/u*(0) and ρ*(z)/ρ*(0) by $\alpha g H^3 \rho_x/48A$ and $\alpha g H^5 \rho_x^2/576AK$ respectively. This procedure yields the following magnitudes:

$$u'(0) = 0.092 \text{ cm sec}^{-1}$$

$$\rho'(0) = 1.15 \times 10^{-7} \text{ gms cm}^{-3}$$

$$F' = 4 \times 10^{-9} \text{ gms cm}^{-2} \text{ sec}^{-1}.$$

The values for u'(0) and $\rho'(0)$ compare well with their respective scale values. The value for the mass flux, F', compares to a value of 10^{-8} gms cm^{-2} sec^{-1} determined from scaling. The $\frac{1}{2}$ order of magnitude difference can be accounted for in the scaling of the vertical derivative of u': if the value $(\frac{3\pi}{2H})\frac{\partial}{\partial z^*}$ had been used which corresponds more closely to the profile in Fig. 2 than the value $(\frac{\pi}{H})\frac{\partial}{\partial z^*}$, the two magnitudes would have been nearly equal. It is suspected, however, that the proposed mechanism will be more important in deeper water where the bottom frictional layer, which accounts for the $\frac{3\pi}{2}$ scaling, is not as significant. Therefore, the scaling will be accepted as it exists; and a search will be made for a range of parameters in which the density-induced flow is significant.

INVESTIGATION OF THE RESIDENCE TIME SCALE FOR THE SIMPLE CASE

It should be noted that the Austauch coefficient has been assumed to be proportional to u_*H with the constant of proportionality being equal to 0.2. Essentially, this assumes that the mixing length scale (or the vertical distance over which turbulence is correlated) is equal to H/5. To generalize the study, it is now assumed that a constant of proportionality β exists which can assume a range of values; then $A = \beta u_* H$.

Using this assumption and reducing the residence time scale, Eq. 18, to the basic parameters gives:

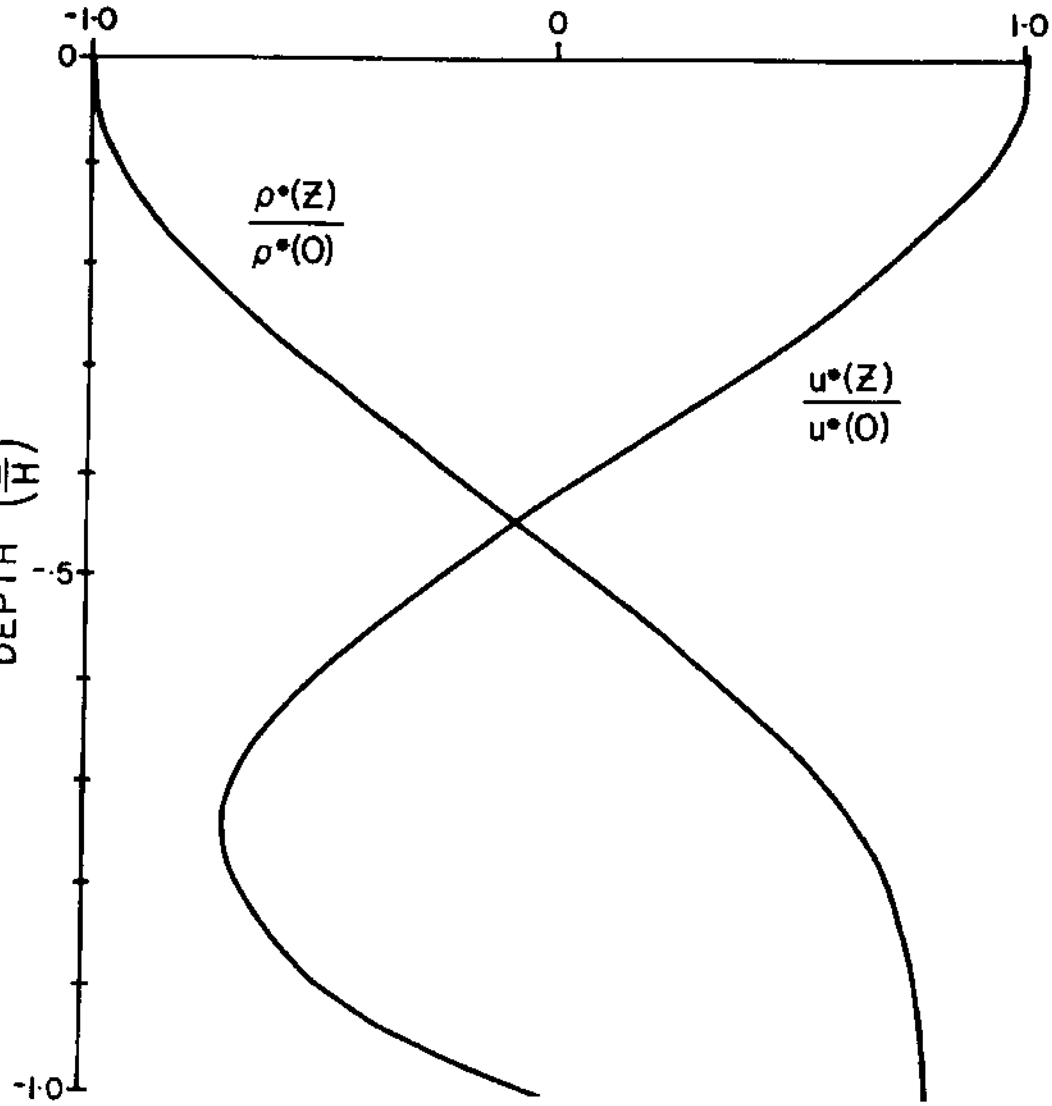

Figure 2. Depth profiles of normalized u* and ρ*

$$Tr = \left(\frac{\pi^7}{\alpha^2 g^2}\right)\left(\frac{L^2\beta^3 u_*^3}{H^5\rho_x^2 Pr}\right) \sim .003\left(\frac{L^2\beta^3 u_*^3}{H^5\rho_x^2 Pr}\right). \qquad (23)$$

The residence time scale can be understood in terms of its basic parameters. The horizontal distance over which advection must occur is L. The longer this distance, the longer it takes to change the horizontal gradient. A measure of vertical exchange is given by βu_*. As vertical exchange increases, the horizontal flow is reduced, which, in turn, increases the residence time. The effect of H can be compared to that of a torquing arm: as H increases, the horizontal density gradient can apply more "leverage" so that the flux increases and Tr diminishes. The basic forcing is ρ_x--the greater this forcing the smaller the residence time. The Prandtl number determines the "short-circuiting" of the system; i.e., for small Pr, the material diffuses rapidly from the outflowing current to the inflowing current and, hence, remains in the system. Then, for a small Pr, the residence time is long.

This estimate of residence time is valid for any conservative material in the water column; Pr then becomes the ratio of momentum diffusion to the diffusion of that material. To determine the range of H and u_* for which this approach is valid, it is necessary to look again at the Ekman number scaling. The assumption was made that $\frac{1}{E_k^2}$ was sufficiently small so that the friction term dominated the Coriolis term. It is reasonable to assume that this approach is valid for $\frac{1}{E_k^2} \leq 0.1$; i.e.,

$$\frac{H^4 f^2}{\pi^4 A^2} \sim \frac{H^2 f^2}{\pi^4 \beta^2 u_*^2} \leq 0.1.$$

Using the scale values for u_*, β and f, then:

$$H \leq .3\pi^2 \frac{\beta u_*}{f} \sim 60 \text{ meters.} \qquad \text{(1st criterion)}$$

An initial and important assumption was made that ρ_x' be much smaller than ρ_x. In scaled notation ρ_x' is estimated as $\frac{H(Ra)\rho_x}{L\Pi}$, where the gradient length scale associated with ρ' is $\frac{1}{L}$. Expand Ra in terms of its basic components and assume that $\rho'_x \leq .1\rho_x$. Then,

$$\frac{gH^3\rho_x(P_r)}{\rho\pi^5\beta^2u_*^2L} \leq 0.1.$$

Using the scaled values for Pr, u_*, β and L, this becomes:

$$H^3\rho_x \leq 150 \text{ gms cm}^{-1}. \qquad \text{(2nd criterion)}$$

Thus, a limit of validity for this approach is established using the two criteria.

Figure 3 shows a plot of residence time as a function of H and ρ_x using the following parameters:

$L = 5 \times 10^5$ cm

$\beta = 0.2$

$u_* = 0.5$ cm sec^{-1}

$Pr = 1.$

Referring to Fig. 3, it is clear that, for the very shallow lagoons of southeast Florida, the horizontal density gradient would have to be extremely large for density-induced flows to be of importance. However, for the deeper bays and for the coastal regions of the ocean, this mechanism could be significant.

Using Eq. 23, it is easy to see how a change of parameters would affect the residence time. For example, doubling the Prandtl number would reduce the residence time by 0.5 and halving β would reduce the residence time by 0.125. Either of these cases are well within the range of possibility.

ACCELERATIONS DURING SLACK TIDE

In the shallow lagoons of southeast Florida, it is clear that the presence of strong mixing is sufficient to nearly eliminate the internal advective motions which might otherwise be induced by horizontal density gradients. If stirring were to suddenly drop toward zero, however, the horizontal density structure

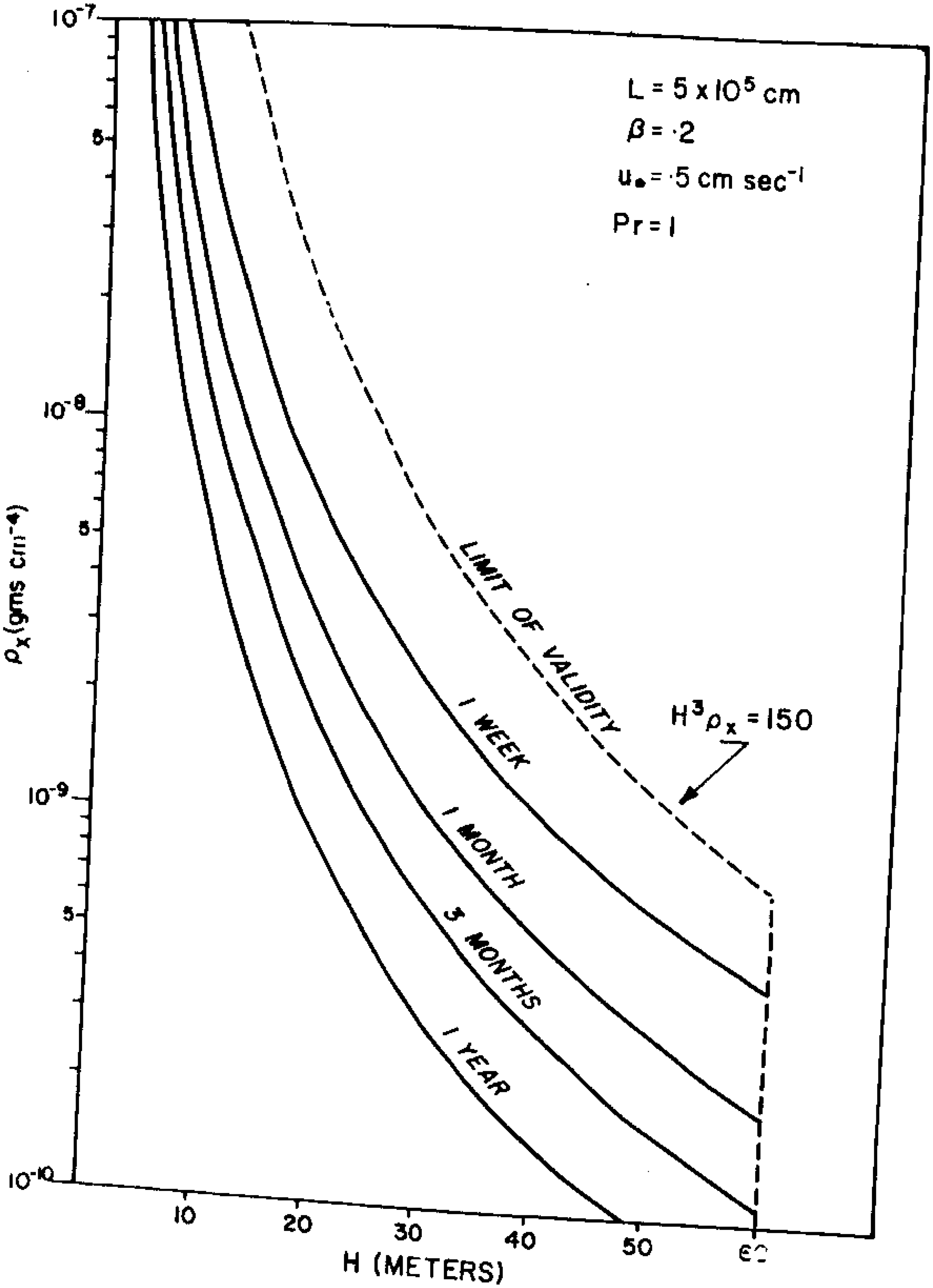

Figure 3. Intralagoon residence time as a function of H and ρ_x

would tend to become unbalanced and to "collapse" in such a fashion as to increase the horizontal flux. This condition may occur during periods of slack tide when the winds are light. The question to be answered is "Does this collapse occur over sufficiently long enough time and with sufficient intensity to substantially decrease the residence time?"

To answer this question, the acceleration terms are included in the system of Eqs. 7-10, and the time derivatives as $\frac{\delta U}{T}$, $\frac{\delta V}{T}$, $\frac{\delta \rho}{T}$, where T is the effective period of slack tide. This period corresponds to the time during which the horizontal density gradient is no longer balanced by vertical diffusion of momentum. The momentum and diffusion equations are now written as:

$$\left(\frac{\pi}{H}\,\frac{\delta U}{T}\right)u^*_{zt} - \left(\frac{\pi f V}{H}\right) v^*_{z} - \left(\frac{\pi^3 A U}{H^3}\right) u^*_{zzz} = \left(\frac{g\rho_x}{\rho}\right)\bar{\rho}^*_x \tag{24}$$

$$\left(\frac{\pi}{H}\,\frac{\delta V}{T}\right)v^*_{zt} + \left(\frac{\pi f U}{H}\right)u^*_{z} - \left(\frac{\pi^3 A V}{H^3}\right)v^*_{zzz} \quad 0 \tag{25}$$

$$\left(\frac{\delta\rho}{T}\right)\rho^*_t + (U\rho_x)\ u^*\bar{\rho}^*_x - \left(\frac{K\pi\rho_z}{H}\right)\rho^*_{zz} = 0. \tag{26}$$

The horizontal friction and diffusion terms have already been seen to be negligible. The vertical friction and diffusion terms are now considered to vary as $(A,K)\sim(A_o,K_o)\ |\sin\sigma t|$, where σ is the angular velocity of the tide.

To determine the period of effective slack tide, the momentum balance is assumed to occur as before (friction balancing pressure) and the derivative in time is taken of this balance and compared to the steady state as follows:

$$U \sim \frac{gH^3\rho_x}{\rho\pi^3 A_o|\sin\sigma t|}$$

$$U_t \sim \frac{gH^3\rho_x\sigma}{\rho\pi^3 A_o}\ \left|\frac{\cot\sigma t}{\sin\sigma t}\right| = U\sigma|\cot\sigma t|.$$

The ratio of acceleration compared to vertical diffusion of momentum, $\frac{\pi^2 AU}{H^2}$, gives:

$$\frac{H^2U\sigma|\cot\sigma t|}{\pi^2A_oU|\sin\sigma t|}\ .$$

The time during which accelerations are dominant occurs when this ratio is greater than unity; or when:

$$\left|\frac{\cot\sigma t}{\sin\sigma t}\right| > \frac{\pi^2A_o}{\sigma H^2}\ .$$

Using the previous scale values and assuming a semidiurnal tide, this gives:

$$\sigma t \ < \ \left|\pm\frac{\pi}{15}\right| \ \text{radians}$$

or a total period of $\frac{1}{15}$ of the tidal cycle; i.e., 50 minutes. Since $f<<\frac{1}{T}$, the Coriolis terms can again be neglected.

The significant balance of terms in Eqs. 24 and 26 will then provide the following estimates of δU and $\delta\rho$:

$$\delta U \sim \frac{gH\rho_xT}{\pi\rho} \sim 1.5 \text{ cm sec}^{-1}$$

$$\delta\rho \sim \frac{gH\rho_x^2T^2}{\pi\rho} \sim 2 \times 10^{-5} \text{ cm sec}^{-1}$$

where T is 50 minutes. The scaled flux of mass during the period of slack tide can be approximated as:

$$F \sim \frac{1}{H}\frac{1}{T}\int_{-H}^{0}\int_{0}^{T}\delta U\delta\rho dtdz \sim \frac{g^2H^2\rho_x^3T^3}{8\pi^2\rho^2}$$

where the modal profiles have been taken into account. Using the previous scales, $F \sim 3.7 \times 10^{-6}$ gm cm^{-2} sec^{-1}. However, since the time duration is only $\frac{1}{15}$ of the tidal cycle, occurring twice during each cycle, this gives an effective flux of about 5×10^{-7} gm cm^{-2} sec^{-1} for the entire cycle or approximately 50 times the estimate for the steady state case. For the shallow lagoons, the residence time is then about 20 years, which is still insignificant as a flushing mechanism.

A simple formula for the residence time, derived in the same manner as the steady state case, is:

$$Tr \sim \frac{8\pi^2\rho^2}{g^2} \frac{L^2}{H^2\rho_x^3\sigma T^4} \sim 8.4 \times 10^{-5} \left(\frac{L^2}{H^2\rho_x^3\sigma T^4}\right)$$

where σ is the angular velocity of the tide and T is the duration of slack tide determined from the relationship:

$$\left|\frac{\cot\sigma t}{\sin\sigma t}\right| > \frac{\pi^2 A_0}{\sigma H^2}.$$

It is doubtful that these estimates can be extended to the deeper water situations since the time it takes for a change in bottom stress to be felt throughout the water column $\left(\sim \frac{H^2}{\pi^2 A}\right)$ would be on the same scale as the duration of slack tide; i.e., residual stirring effects may be present during slack tide.

SOLUTION OF THE ACCELERATION CASE

Using the results of scaling, the equations for motion and density during the period of slack tide are:

$$u^*_{zt} = \bar{\rho}^*_x \tag{27}$$

$$\rho^*_t = u^*\bar{\rho}^*_x \tag{28}$$

with the conversion scales:

$$u' = \left(\frac{gH\rho_x T}{\rho}\right) u^*$$

$$\rho' = \left(\frac{gH\rho_x^2 T^2}{\rho}\right) \rho^*$$

where the π's have been omitted since they are profile shaping factors. The boundary conditions are:

(I) $$u^*(z,t=0) = -\gamma_1 \frac{\bar{\rho}_x^*}{8}\left(\frac{4z^3}{3} + \frac{3z^2}{2} - \frac{1}{6}\right)$$

(II) $$\int_{-1}^{0} u^*(z,t)dz=0$$

(III) $$\rho^*(z,t=0) = -\gamma_2 \frac{(\bar{\rho}_x^*)^2}{8}\left(\frac{z^5}{15} + \frac{z^4}{8} - \frac{z^2}{12} + \frac{1}{72}\right)$$

where γ_1 and γ_2 are the ratios of the steady state conversion scales to the acceleration case conversion scales for u^* and ρ^* respectively:

$$\gamma_1 = \frac{H^2}{AT}$$

$$\gamma_2 = Pr\ \gamma_1^2 .$$

Pr is the Prandtl number.

The solutions to Eqs. 27 and 28, using boundary conditions (I) – (III) are:

$$u^* = \bar{\rho}_x^*\ t^*\ (z + \tfrac{1}{2}) - \gamma_1 \frac{\bar{\rho}_x^*}{8}\left(\frac{4z^3}{2} + \frac{3z^2}{2} - \frac{1}{6}\right) \tag{29}$$

$$\rho^* = -(\bar{\rho}_x^*)^2 \left[(z + \tfrac{1}{2})\frac{t^{*2}}{2} - \frac{\gamma_1}{8}\left(\frac{4}{3}z^3 + \frac{3}{z}z^2 - \frac{1}{6}\right)t^* + \frac{\gamma_2}{8}\left(\frac{z^5}{15} + \frac{z^4}{8} - \frac{z^2}{12} + \frac{1}{72}\right)\right]. \tag{30}$$

The time variable, t^*, is normalized by the period of effective slack tide; i.e.,

$$t^* = \frac{t}{T} .$$

Eqs. 29 and 30 are plotted against depth for selected time intervals and presented in Figs. 4 and 5, respectively. Figure 6 shows a plot of the density flux as a function of time. The average density flux for the

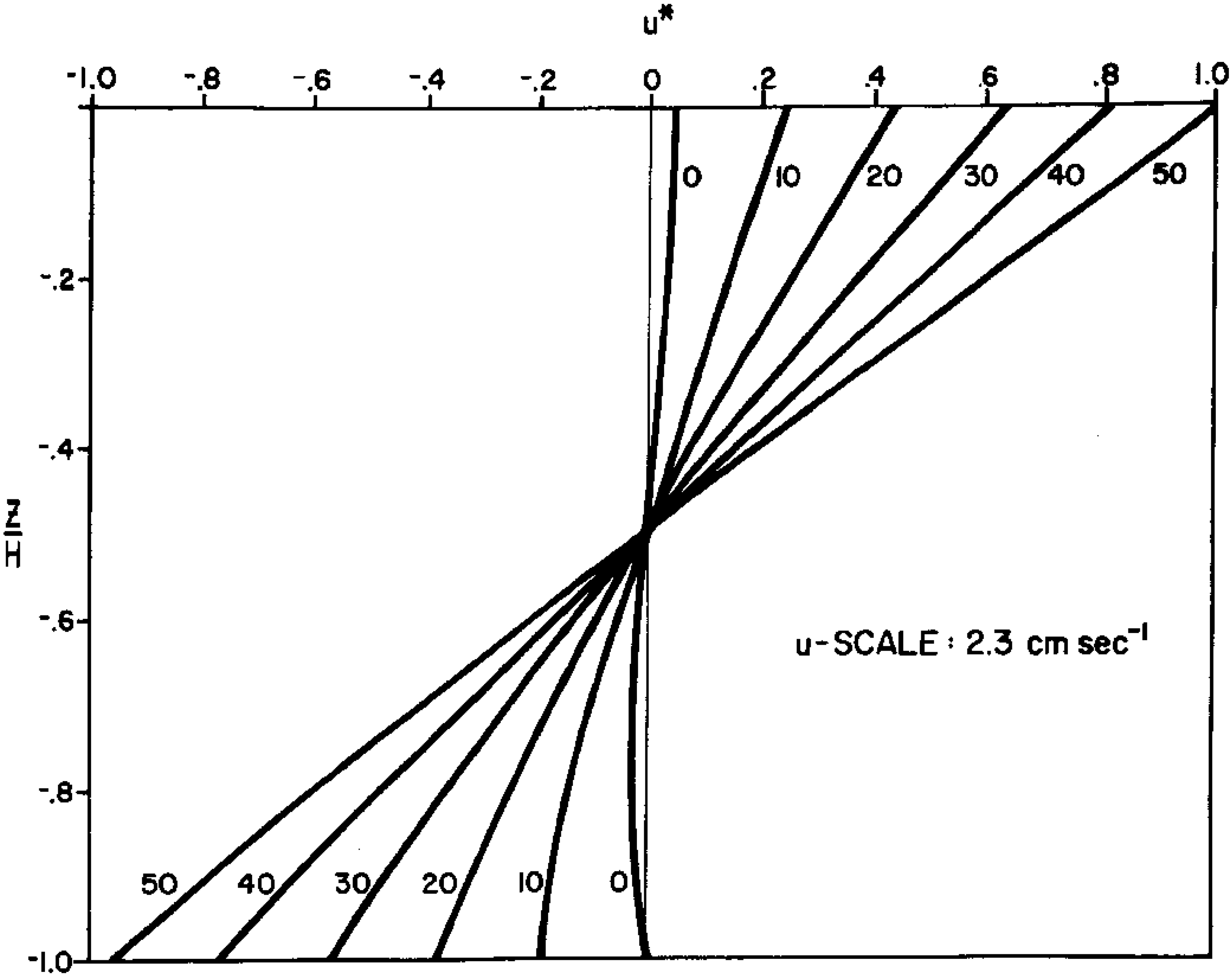

Figure 4. Normalized depth profile of u* as a function of time (in minutes) from Eq. 29

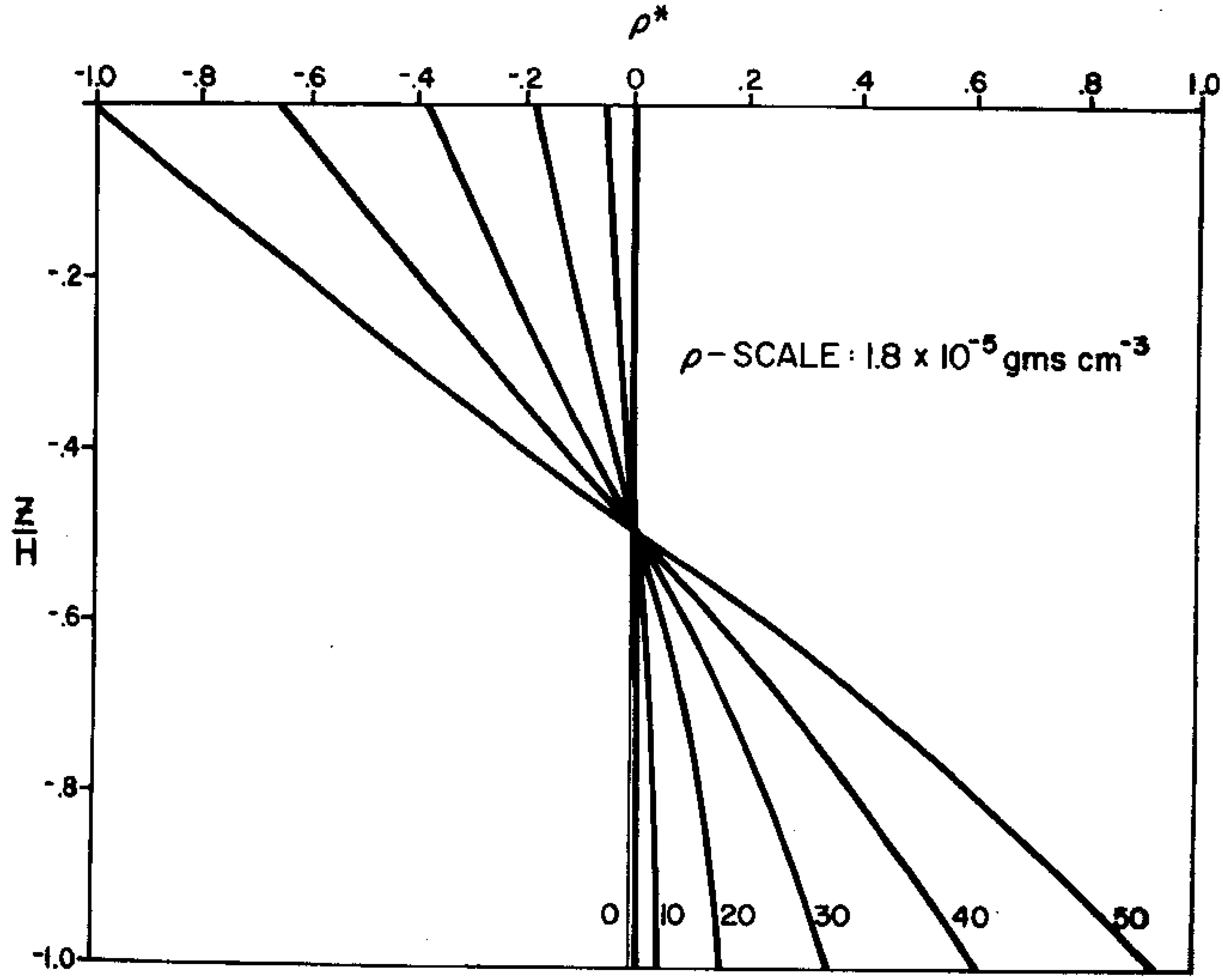

Figure 5. Normalized depth profile of ρ^* as a function of time (in minutes) from Eq. 30

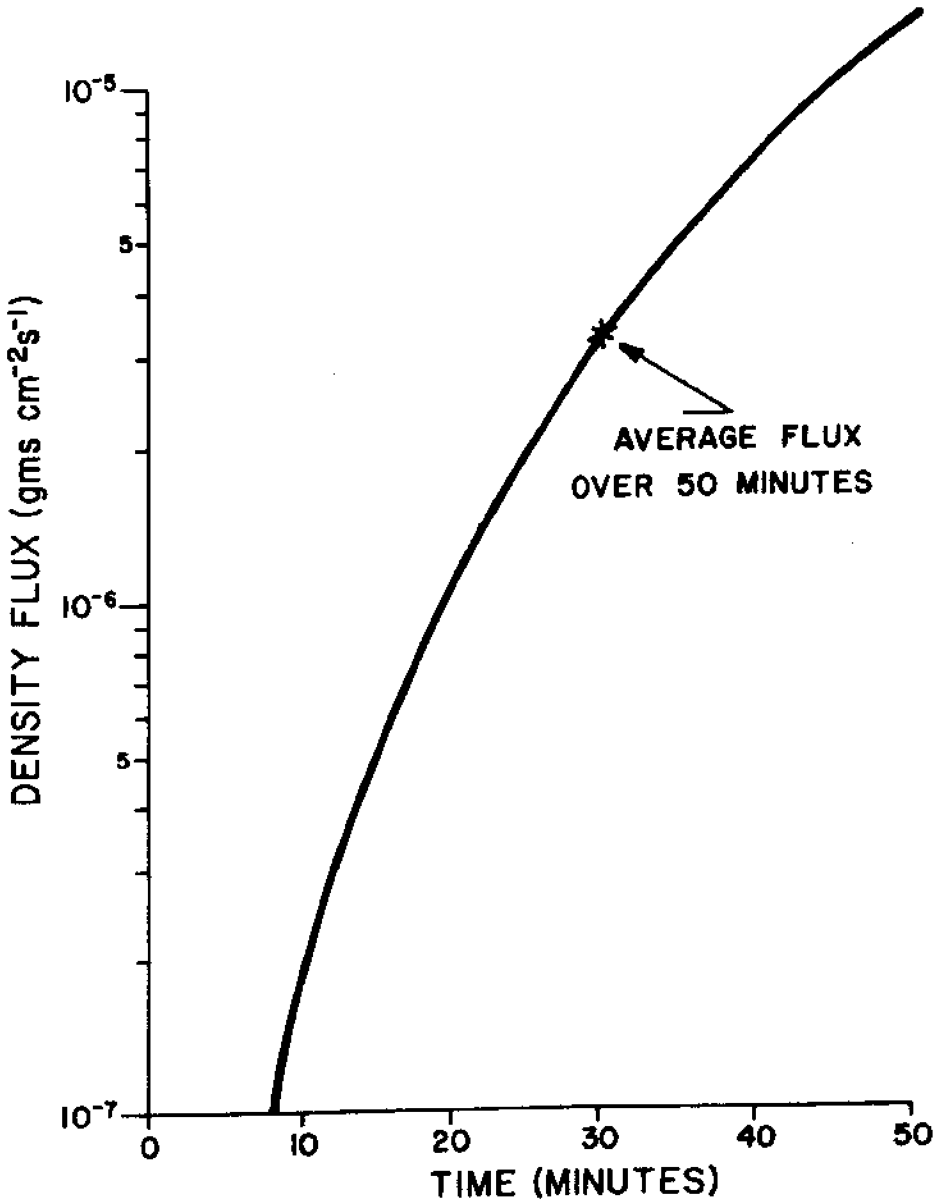

Figure 6. Density flux as a function of time (minutes)

duration of slack tide is calculated as 3.72×10^{-6} gms cm^{-2} sec^{-1} which tends to verify the scaling relations.

The conclusion remains the same: for the shallow lagoons of southeast Florida density-induced accelerations during slack tide are not sufficient to affect the residence time.

SUMMARY AND CONCLUSIONS

Using simple models, the influence of a horizontal density gradient on the residence time of a shallow, well-stirred lagoon was investigated. Under steady state conditions and with scaling taken from the southeast Florida lagoons (Biscayne Bay and Card Sound), it was found that residence time resulting solely from density-induced motions was on the order of 1,000 years. If the stirring were provided only by the tides, it was determined that during slack tide the available potential energy contained in the horizontal density gradient was converted to kinetic energy; and the resulting advective motions reduced the residence time to about 20 years. However, the best estimate of the actual residence time produced by the interaction of wind-, tide-, and density-induced motions is approximately 3 months.

Thus, it can be concluded that density-induced motions do not contribute substantially to the flushing of the lagoons. In both cases, it is the shallowness of the lagoons and the uniformity of conditions which diminish the horizontal advective motions. It was also shown by Lee and Rooth (3, 4) that flushing of interior lagoon waters by tidal processes is very weak, resulting in residence times on the order of a year. Therefore, it is believed that wind-generated circulations are the dominant renewal mechanisms responsible for the 3-month residence time of these shallow embayments.

Although horizontal density gradient-induced flows were found to be insignificant in flushing Biscayne Bay and Card Sound, there are many estuaries where this mechanism may be important. For instance, Eq. 23 and Fig. 3 show that as depth increases, the horizontal density gradient can exert more "leverage," thus increasing the flux and decreasing the residence time. Also, as the basic forcing (density gradient) increases or the horizontal length scale (L) of the density gradient decreases, the residence time will decrease, both of which can occur in estuaries with narrow frontal zones.

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APPENDIX I. - REFERENCES

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APPENDIX II. - NOTATION

x, y, z = rectangular Cartesian coordinates: x positive to the east; y positive to the north; z positive upwards;

f = Coriolis parameter;

u, v, w = velocity components in the x, y, z directions, respectively;

A, A_h = vertical and horizontal (x-component) Austach coefficients;

K, K_h = vertical and horizontal (x-component) diffusion coefficients, respectively;

p = pressure;

ρ = density;

g = acceleration of gravity;

α = specific volume;

$\overline{u}, \overline{v}, \overline{w}, \overline{\rho}$ = velocity components and density as a function of x only;

u', v', w', ρ' = velocity components and density as a function of x and z;

H = mean water depth;

ψ = stream function;

m = modal number;

$*$ = symbol for scaled variable (non-dimensional);

U, V, W = dimensional form of velocity components;

L = horizontal length scale;

k = Karmen coeffecient;

u_* = friction velocity;

τ = total stress;

P_r = Prandtl number;

E_k = Ekman number;

R_a = Rayleigh number;

F' = total mass flux;

F = scaled mass flux;

T_r = residence time scale;

β = constant of proportionality;

δ = symbol for time derivative;

T = effective period of slack tide;

σ = angular tidal velocity;

A_o, K_o = tidal amplitude of the vertical Austach and diffusion coefficients, respectively;

t = time; and

γ_1, γ_2 = ratios of the steady state conversion scales to acceleration case conversion scales for u^* and ρ^*, respectively.